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CONTROL PROBLEMS FOR THE STEADY-STATE EQUATIONS OF MAGNETOHYDRODYNAMICS OF A VISCOUS INCOMPRESSIBLE FLUID

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Control problems for a steady-state model of the magnetohydrodynamics of a viscous incompressible fluid in a bounded domain with an impermeable, perfectly conducting boundary are formulated. The resolvability of the problems is studied, the use of the Lagrange principle is justified, and optimality systems are analyzed.

Key words: *magnetohydrodynamics, viscous fluid, control problems.*

The problems of controlling viscous conducting flows play an important role in some applied areas of magnetohydrodynamics, including the development of magnetohydrodynamic (MHD) generators, the design of new underwater engines, the modeling of nuclear reactor cooling systems, and nuclear fusion control [1, 2]. The development of methods and algorithms for solving the indicated problems has been the subject of much research. Theoretical issues related to analysis of the resolvability and properties of control problems have been studied less extensively. The present paper considers the indicated problems for the MHD-model of a viscous fluid.

1. Formulation of the Boundary-Value Problem. Let Ω be a bounded domain with an impermeable, perfectly conducting boundary Γ. We consider the boundary-value problem for the steady-state equations of magnetohydrodynamics of a viscous fluid in the domain Ω. In dimensionless variables, this problem has the form

$$-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \mu \operatorname{rot}\boldsymbol{H}\times\boldsymbol{H} = \boldsymbol{f}, \qquad \operatorname{div}\boldsymbol{u} = 0; \tag{1.1}$$

$$\nu_m \operatorname{rot}\boldsymbol{H} - \boldsymbol{E} + \boldsymbol{H}\times\boldsymbol{u} = \boldsymbol{E}_0, \qquad \operatorname{div}\boldsymbol{H} = 0, \qquad \operatorname{rot}\boldsymbol{E} = 0; \tag{1.2}$$

$$\boldsymbol{u} = 0 \quad \text{on} \quad \Gamma, \qquad \boldsymbol{H}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{E}\times\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma. \tag{1.3}$$

Here $\boldsymbol{u}$, $\boldsymbol{H}$, and $\boldsymbol{E}$ are the velocity and the electric- and magnetic-field strength vectors, respectively, p is the pressure, $\nu = 1/\mathrm{Re}$, $\nu_m = 1/\mathrm{Re}_m$ (Re is the Reynolds number and Re_m is the magnetic Reynolds number), μ is a dimensionless parameter, $\boldsymbol{f}$ is the dimensionless vector of the external body force vector, $\boldsymbol{E}_0$ is the strength of extraneous forces (for the physical meaning and properties of the function $\boldsymbol{E}_0$ see in greater detail in [3, p. 47]).

The goal of the present study is to formulate and study control problems for the model (1.1)–(1.3). To formulate the control problem, we divide the initial data, i.e., the pair $(\boldsymbol{f}, \boldsymbol{E}_0)$, into two groups: a group of rigid (unchanged) data and a group of controls. The first group includes the density $\boldsymbol{f}$, and the group of controls includes the function $\boldsymbol{E}_0$, which is considered unknown and is chosen from the minimum condition for a particular quality functional (see Sec. 3).

To investigate the thus formulated control problem, we employ the procedure designed in [4–6] for control problems based on steady-state heat- and mass-transfer models. This procedure reduces control problems to problems belonging to the class of conditional extremum problems in Hilbert spaces, in which the main constraint is a weak (in the sense of generalized functions) formulation of the model. A theory for solving abstract problems of this type has been developed (see, for example, [7]). This approach considerably simplifies the derivation of an

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optimality system and reduces construction of this system to verification of a number of conditions that ensure validity of the indicated theory. For our problem, such conditions are the continuous differentiability of the quality functional and the operators of the examined model with respect to state variables (velocity, pressure, and temperature), which is usually the case, and the convex dependence of the parameters on controlling parameters, which, as a rule, is implied. The main condition to be verified is the Fredholm property of the linear operator equal to the Frechet derivative of the operator of the examined model with respect to the state.

We note that for a simplified MHD-model (in which the strength $\boldsymbol{H}$ is considered specified), a control problem similar in formulation is studied in [8], where the control was the electric current component normal to the boundary.

In constructing the corresponding theory for the model (1.1)–(1.3), we use Sobolev's spaces $H^s(D)$ and $L^2(\Omega) \equiv H^0(\Omega)$ and their vector analogs $\boldsymbol{H}^s(D)$ ($s \in \mathbb{R}$), where D means Ω or Γ. The scalar products in $L^2(\Omega)$ and $\boldsymbol{L}^2(\Omega)$ are designated by $(\,\cdot\,,\,\cdot\,)$; the norm in $L^2(\Omega)$ by $\|\cdot\|$; the norm or seminorm in $H^1(\Omega)$ and $\boldsymbol{H}^1(\Omega)$, by $\|\cdot\|_1$ or $|\cdot|_1$; and the duality ratio for the pair X and X^* by $\langle\,\cdot\,,\,\cdot\,\rangle_{X^*\times X}$ or $\langle\,\cdot\,,\,\cdot\,\rangle$. We set $L^2_0(\Omega) = \{p \in L^2(\Omega)\colon (p, 1) = 0\}$, $H^1_0(\Omega) = \{\varphi \in H^1(\Omega)\colon \varphi g|_\Gamma = 0\}$, $\boldsymbol{H}^1_0(\Omega) = H^1_0(\Omega)^3$, $\boldsymbol{H}^1_T(\Omega) = \{\boldsymbol{h} \in \boldsymbol{H}^1(\Omega)\colon \boldsymbol{h}\cdot\boldsymbol{n}|_\Gamma = 0\}$, $\boldsymbol{H}(\mathrm{rot};\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\colon \mathrm{rot}\,\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}$, and $\boldsymbol{H}^1(\Delta;\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)\colon \Delta\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}$. Here, in particular, $\boldsymbol{H}^1_T(\Omega)$ consists of vectors $\boldsymbol{h} \in \boldsymbol{H}^1(\Omega)$ tangential on Γ. A detail description of the properties of these spaces is given in [9–11]. Let the following conditions be satisfied:

Condition 1. Ω is a convex polyhedron or a bounded, finitely connected, domain in space $\mathbb{R}^3$ with boundary $\Gamma \in C^{1,1}$, consisting of $p_0 + 1$ connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_{p_0}$ (Γ_0 is the boundary of the infinite component of the set $\mathbb{R}^3 \setminus \overline{\Omega}$), and there exist surfaces $\Sigma_i \in C^2$ ($i = 1, 2, \dots, q_0$) such that $\Sigma_i \cap \Sigma_j = \varnothing$ for $i \neq j$; the set $\tilde{\Omega} = \Omega \setminus \bigcup\limits_{i=1}^{q_0} \Sigma_i$ is simply connected and Lipschitzian.

Condition 2. $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$.

Condition 3. $\boldsymbol{j} \in \boldsymbol{L}^2(\Omega)$.

The numbers q_0 and p_0 appearing in Condition 1 are referred to as the first and second Betti numbers, respectively. They are topological characteristics of the domain Ω; $p_0 = 0$ if and only if the boundary Γ is connected, and $q_0 = 0$ if and only if the domain Ω is simply connected. Let us introduce two spaces of vectors $\mathcal{H}(e)$ and $\mathcal{H}(m)$ which are harmonic in Ω and consist of solutions of homogeneous problems of electric and magnetic type:

$$\mathrm{div}\,\boldsymbol{E} = 0, \quad \mathrm{rot}\,\boldsymbol{E} = 0 \quad \text{in} \quad \Omega, \qquad \boldsymbol{E}\times\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma,$$

$$\mathrm{div}\,\boldsymbol{H} = 0, \quad \mathrm{rot}\,\boldsymbol{H} = 0 \quad \text{in} \quad \Omega, \qquad \boldsymbol{H}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma.$$

As is known, the spaces $\mathcal{H}(e)$ and $\mathcal{H}(m)$ are finite-dimensional; $\dim\mathcal{H}(e) = p_0$ and $\dim\mathcal{H}(m) = q_0$ [12]. Let us designate the orthogonal addition to $\mathcal{H}(m)$ in $\boldsymbol{L}^2(\Omega)$ by $\mathcal{H}(m)^\perp$.

In the study of the problem, of key importance are the spaces $\boldsymbol{H}^1_0(\Omega)$, $\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)\colon \mathrm{div}\,\boldsymbol{v} = 0\}$, $\boldsymbol{V}_T = \{\boldsymbol{h} \in \boldsymbol{H}^1_T(\Omega) \cap \mathcal{H}(m)^\perp\colon \mathrm{div}\,\boldsymbol{h} = 0\}$, the product $X = \boldsymbol{V}\times\boldsymbol{V}_T$, and their dual spaces $\boldsymbol{H}^{-1}(\Omega) = (\boldsymbol{H}^1_0(\Omega))^*$, $\boldsymbol{V}^*$, $\boldsymbol{V}^*_T$, and $X^* = \boldsymbol{V}^*\times\boldsymbol{V}^*_T$. Each of the spaces $\boldsymbol{H}^1_0(\Omega)$, $\boldsymbol{V}$, and $\boldsymbol{V}_T$ is a Hilbert space with the norm $\|\cdot\|_1$. The space X is a Hilbert one with the norm $(\boldsymbol{v},\boldsymbol{h}) \to \|(\boldsymbol{v},\boldsymbol{h})\|_1 = (\|\boldsymbol{v}\|_1^2 + \|\boldsymbol{h}\|_1^2)^{1/2}$. The following orthogonal expansion is valid [12]:

$$\boldsymbol{L}^2(\Omega) = \mathrm{rot}\,\boldsymbol{V}_T \oplus \nabla H^1_0(\Omega) \oplus \mathcal{H}(e). \tag{1.4}$$

It implies that any vector $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ is represented in a single fashion as $\boldsymbol{f} = \mathrm{rot}\,\boldsymbol{q} + \nabla\varphi + \boldsymbol{e}$. Here the vector potential $\boldsymbol{q} \in \boldsymbol{V}_T$, the scalar potential $\varphi \in H^1_0(\Omega)$, and the harmonic vector $\boldsymbol{e} \in \mathcal{H}(e)$ I are uniquely determined from $\boldsymbol{f}$.

Let us introduce bilinear and trilinear forms:

$$a_0(\boldsymbol{u},\boldsymbol{v}) = \int\limits_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v}\, d\Omega, \qquad a_1(\boldsymbol{H},\boldsymbol{\Psi}) = \int\limits_\Omega \mathrm{rot}\,\boldsymbol{H}\cdot\mathrm{rot}\,\boldsymbol{\Psi}\, d\Omega,$$
$$c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \equiv \int\limits_\Omega [(\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{v}]\cdot\boldsymbol{w}\, d\Omega, \qquad c_1(\boldsymbol{\Psi},\boldsymbol{H},\boldsymbol{u}) = \int\limits_\Omega (\mathrm{rot}\,\boldsymbol{\Psi}\times\boldsymbol{H})\cdot\boldsymbol{u}\, d\Omega. \tag{1.5}$$

If Condition 1 is satisfied, the introduced forms are continuous and the forms a_0 and a_1 are in addition coercive in the spaces $\boldsymbol{V}$ and $\boldsymbol{V}_T$, respectively [11, 12]. There exist constants C_1, α_i, γ_i, and γ_i' ($i = 0, 1$) that depend on Ω and satisfy the following inequalities:

$$|a_0(\boldsymbol{u},\boldsymbol{v})| \leqslant \|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1 \quad \forall(\boldsymbol{u},\boldsymbol{v}) \in \boldsymbol{H}^1(\Omega)^2, \qquad a_0(\boldsymbol{v},\boldsymbol{v}) \geqslant \alpha_0\|\boldsymbol{v}\|^2 \quad \forall \boldsymbol{v} \in \boldsymbol{V}; \tag{1.6}$$

$$|a_1(\boldsymbol{H},\boldsymbol{\Psi})| \leqslant C_1^2\|\boldsymbol{H}\|_1\|\boldsymbol{\Psi}\|_1 \quad \forall(\boldsymbol{H},\boldsymbol{\Psi}) \in \boldsymbol{H}^1(\Omega)^2, \qquad a_1(\boldsymbol{\Psi},\boldsymbol{\Psi}) \geqslant \alpha_1\|\boldsymbol{\Psi}\|_1^2 \quad \forall \boldsymbol{\Psi} \in \boldsymbol{V}_T; \tag{1.7}$$

$$|c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leqslant \gamma'\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_{\boldsymbol{L}^4(\Omega)} \leqslant \gamma\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1 \qquad \forall(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{H}^1(\Omega)^3; \tag{1.8}$$

$$|c_1(\boldsymbol{\Psi},\boldsymbol{H},\boldsymbol{v})| \leqslant \gamma_1'\|\boldsymbol{\Psi}\|_1\|\boldsymbol{H}\|_1\|\boldsymbol{v}\|_{\boldsymbol{L}^4(\Omega)} \leqslant \gamma_1\|\boldsymbol{\Psi}\|_1\|\boldsymbol{H}\|_1\|\boldsymbol{v}\|_1 \qquad \forall(\boldsymbol{\Psi},\boldsymbol{H},\boldsymbol{v}) \in \boldsymbol{H}^1(\Omega)^3. \tag{1.9}$$

In addition, it is known (see, for example, [9]) that

$$c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0, \quad c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -c(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) \quad \forall \boldsymbol{u} \in \boldsymbol{V}, \quad \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \quad \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega). \tag{1.10}$$

We set $\nu_1 = \mu\nu_m$, $a((\boldsymbol{u},\boldsymbol{H}),(\boldsymbol{v},\boldsymbol{\Psi})) \equiv \nu a_0(\boldsymbol{u},\boldsymbol{v}) + \nu_1 a_1(\boldsymbol{H},\boldsymbol{\Psi})$, and $\lambda_* = \min(\alpha_0\nu, \alpha_1\nu_1)$. By virtue of (1.6) and (1.7), the form a is coercive in the space X with constant λ_*, so that

$$a((\boldsymbol{v},\boldsymbol{\Psi}),(\boldsymbol{v},\boldsymbol{\Psi})) \geqslant \lambda_*\|(\boldsymbol{v},\boldsymbol{\Psi})\|_1^2 \equiv \lambda_*(\|\boldsymbol{v}\|_1^2 + \|\boldsymbol{\Psi}\|_1^2) \qquad \forall(\boldsymbol{v},\boldsymbol{\Psi}) \in X. \tag{1.11}$$

Below, we use the following Green formulas:

$$(\boldsymbol{u}, \operatorname{grad}\varphi) = -(\operatorname{div}\boldsymbol{u}, \varphi) \qquad \forall \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega), \quad \varphi \in H^1(\Omega); \tag{1.12}$$

$$(\operatorname{rot}\boldsymbol{q}, \boldsymbol{w}) - (\boldsymbol{q}, \operatorname{rot}\boldsymbol{w}) = -\langle \boldsymbol{q} \times \boldsymbol{n}, \boldsymbol{w}\rangle_\Gamma \qquad \forall \boldsymbol{q} \in \boldsymbol{H}(\mathrm{rot};\Omega), \quad \boldsymbol{w} \in \boldsymbol{H}^1(\Omega); \tag{1.13}$$

$$-(\Delta\boldsymbol{u}, \boldsymbol{v}) = (\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) \equiv \int\limits_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v}\, d\Omega \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^1(\Delta;\Omega), \quad \boldsymbol{v} \in H_0^1(\Omega). \tag{1.14}$$

Here $\boldsymbol{q} \times \boldsymbol{n} \in \boldsymbol{H}^{-1/2}(\Gamma)$ is the tangential trace of the function $\boldsymbol{q} \in \boldsymbol{H}(\mathrm{rot};\Omega)$ and $\langle \boldsymbol{q} \times \boldsymbol{n}, \boldsymbol{w}\rangle_\Gamma$ is the value of the functional $\boldsymbol{q} \times \boldsymbol{n}$ on the element $\boldsymbol{w}|_\Gamma \in \boldsymbol{H}^{1/2}(\Gamma)$.

We note that the resolvability of the problem (1.1)–(1.3) has been explored in a number of papers [13, 14]. However, we cannot use the results of these studies because they were performed for different functional spaces and different conditions on the initial data. Therefore, we first explore the resolvability of the problem (1.1)–(1.3) (Problem 1).

2. Examination of the Problem 1. We shall introduce an element $\boldsymbol{F} \in X^*$: $\langle \boldsymbol{F}, (\boldsymbol{v},\boldsymbol{\Psi})\rangle = (\boldsymbol{f},\boldsymbol{v}) + \mu(\boldsymbol{E}_0, \operatorname{rot}\boldsymbol{\Psi})$. Obviously,

$$\|\boldsymbol{F}\|_{X^*} \leqslant M \equiv \|\boldsymbol{f}\| + \mu\|\boldsymbol{E}_0\|. \tag{2.1}$$

We multiply the first equation in (1.1) by $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ and the first equation in (1.2) by $\mu \operatorname{rot}\boldsymbol{\Psi}$ ($\boldsymbol{\Psi} \in \boldsymbol{V}_T$), integrate over Ω, and employ formulas (1.12)–(1.14). Taking into account that $(\boldsymbol{E}, \operatorname{rot}\boldsymbol{\Psi}) = 0$ by virtue of (1.13) and the conditions $\operatorname{rot}\boldsymbol{E} = 0$ in Ω and $\boldsymbol{E} \times \boldsymbol{n} = 0$ on Γ, we obtain

$$\nu \int\limits_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v}\, d\Omega + \int\limits_\Omega [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\boldsymbol{v}\, d\Omega - \mu \int\limits_\Omega (\operatorname{rot}\boldsymbol{H} \times \boldsymbol{H})\cdot\boldsymbol{v}\, d\Omega - \int\limits_\Omega p \operatorname{div}\boldsymbol{v}\, d\Omega = \int\limits_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, d\Omega; \tag{2.2}$$

$$\nu_1 \int\limits_\Omega \operatorname{rot}\boldsymbol{H}\cdot\operatorname{rot}\boldsymbol{\Psi}\, d\Omega + \mu \int\limits_\Omega (\boldsymbol{H} \times \boldsymbol{u})\cdot\operatorname{rot}\boldsymbol{\Psi}\, d\Omega = \mu \int\limits_\Omega \boldsymbol{E}_0 \cdot \operatorname{rot}\boldsymbol{\Psi}\, d\Omega. \tag{2.3}$$

Summing up the restriction of the identity (2.2) on the space $\boldsymbol{V}$ and (2.3), we arrive at the weak formulation of the Problem 1: to find a pair $(\boldsymbol{u},\boldsymbol{H}) \in X \equiv \boldsymbol{V} \times \boldsymbol{V}_T$ that satisfies the following identity, written with the notation of Sec. 1:

$$a((\boldsymbol{u},\boldsymbol{H}),(\boldsymbol{v},\boldsymbol{\Psi})) + c(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + \mu[c_1(\boldsymbol{\Psi},\boldsymbol{H},\boldsymbol{u}) - c_1(\boldsymbol{H},\boldsymbol{H},\boldsymbol{v})] = \langle \boldsymbol{F}, (\boldsymbol{v},\boldsymbol{\Psi})\rangle \quad \forall(\boldsymbol{v},\boldsymbol{\Psi}) \in X. \tag{2.4}$$

It should be noted that although the identity (2.4) do not include the pressure p and the electric field $\boldsymbol{E}$, these quantities can be found from a pair $(\boldsymbol{u},\boldsymbol{H}) \in X$ that satisfies (2.4). Indeed, setting $\boldsymbol{\Psi} = 0$ in (2.4), we have $\nu a_0(\boldsymbol{u},\boldsymbol{v}) + c(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - \mu c_1(\boldsymbol{H},\boldsymbol{H},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v})$ $\forall \boldsymbol{v} \in \boldsymbol{V}$. Let us introduce a functional $\boldsymbol{L} = \boldsymbol{L}(\boldsymbol{u},\boldsymbol{H},\boldsymbol{f})$ in $\boldsymbol{H}_0^1(\Omega)$ that acts by the formula $\langle \boldsymbol{L}, \boldsymbol{v}\rangle = \nu a_0(\boldsymbol{u},\boldsymbol{v}) + c(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - \mu c_1(\boldsymbol{H},\boldsymbol{H},\boldsymbol{v}) - (\boldsymbol{f},\boldsymbol{v})$. Obviously, $\boldsymbol{L} \in \boldsymbol{H}^{-1}(\Omega)$ and the restriction of $\boldsymbol{L}$ on $\boldsymbol{V}$ satisfies the condition $\langle \boldsymbol{L}, \boldsymbol{v}\rangle = 0$ $\forall \boldsymbol{v} \in \boldsymbol{V}$. In this case, from [10, p. 22] it follows that there exist a function (pressure) $p \in L_0^2(\Omega)$ such that the identity (2.2), which is equivalent to Eq. (1.1) in the sense of generalized functions, is satisfied.

Setting $\boldsymbol{v} = 0$ in (2.4), we arrive at the identity (2.3), which implies that the vector $\nu_m \operatorname{rot} \boldsymbol{H} + \boldsymbol{H} \times \boldsymbol{u} - \boldsymbol{E}_0$ is orthogonal to $\operatorname{rot} \boldsymbol{\Psi}$ ($\boldsymbol{\Psi} \in \boldsymbol{V}_T$ is an arbitrary vector function). By virtue of (1.4), this is possible if and only if $\nu_m \operatorname{rot} \boldsymbol{H} + \boldsymbol{H} \times \boldsymbol{u} - \boldsymbol{E}_0 = \operatorname{grad} \varphi + \boldsymbol{e}$. Here $\varphi \in H_0^1(\Omega)$ is a scalar potential and $\boldsymbol{e} \in \mathcal{H}(e)$ is a certain vector. We set $\boldsymbol{E} = \operatorname{grad} \varphi + \boldsymbol{e}$. Then, with allowance for the conditions $\operatorname{rot} \boldsymbol{e} = 0$, $\boldsymbol{e} \times \boldsymbol{n}|_\Gamma = 0$, and $\varphi|_\Gamma = 0$, we obtain $\operatorname{rot} \boldsymbol{E} = 0$ and $\boldsymbol{E} \times \boldsymbol{n}|_\Gamma = 0$; in this case, the triple $(\boldsymbol{u}, \boldsymbol{H}, \boldsymbol{E})$ satisfies all relations in (1.2).

Note that a pair $(\boldsymbol{u}, \boldsymbol{H}) \in \boldsymbol{V} \times \boldsymbol{V}_T$ that satisfies (2.4) uniquely determines the pressure $p \in L_0^2(\Omega)$ and the electric field $\boldsymbol{E}$. Hence, we can correctly introduce the following definition.

Definition 2.1. Any pair $(\boldsymbol{u}, \boldsymbol{H}) \in X \equiv \boldsymbol{V} \times \boldsymbol{V}_T$ that satisfies the identity (2.4) is called a weak solution of the Problem 1.

To prove the existence of a solution $(\boldsymbol{u}, \boldsymbol{H}) \in X$ of the problem (2.4), in the space X we introduce an operator G: $X \to X$ that acts by the formula $G(\boldsymbol{w}, \boldsymbol{h}) = (\boldsymbol{u}, \boldsymbol{H}) \in X$. Here the pair $(\boldsymbol{u}, \boldsymbol{H}) \in X$ is a solution of the linear problem

$$a((\boldsymbol{u}, \boldsymbol{H}), (\boldsymbol{v}, \boldsymbol{\Psi})) + a_{\boldsymbol{w},\boldsymbol{h}}((\boldsymbol{u}, \boldsymbol{H}), (\boldsymbol{v}, \boldsymbol{\Psi})) = \langle \boldsymbol{F}, (\boldsymbol{v}, \boldsymbol{\Psi}) \rangle \qquad \forall (\boldsymbol{v}, \boldsymbol{\Psi}) \in X \tag{2.5}$$

obtained by linearization of the nonlinear problem (2.4), where

$$a_{\boldsymbol{w},\boldsymbol{h}}((\boldsymbol{u}, \boldsymbol{H}), (\boldsymbol{v}, \boldsymbol{\Psi})) = c(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) + \mu[c_1(\boldsymbol{\Psi}, \boldsymbol{h}, \boldsymbol{u}) - c_1(\boldsymbol{H}, \boldsymbol{h}, \boldsymbol{v})]. \tag{2.6}$$

Clearly, the term $c_1(\boldsymbol{\Psi}, \boldsymbol{h}, \boldsymbol{u})$ in (2.6) obtained by linearization of the convective term $c_1(\boldsymbol{\Psi}, \boldsymbol{H}, \boldsymbol{u})$ in (2.4), which has the meaning of Maxwell's advective term, and the term $-c_1(\boldsymbol{H}, \boldsymbol{h}, \boldsymbol{v})$ obtained by linearization of the term containing the Lorentzian force $-c_1(\boldsymbol{H}, \boldsymbol{H}, \boldsymbol{v})$ and occurring in (2.4) cancel for $(\boldsymbol{u}, \boldsymbol{H}) = (\boldsymbol{v}, \boldsymbol{\Psi})$. In addition, because $c(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{v}) = 0$ for $\boldsymbol{v} \in \boldsymbol{V}$ by virtue of (1.10), the coercivity of the main form a on X, which follows from (1.11), implies the coercivity of the sum of forms $a + a_{\boldsymbol{w},\boldsymbol{h}}$ in (2.5) with the same constant λ_*. In this case, the Lax–Milgram theorem [10] implies that for any pair $(\boldsymbol{w}, \boldsymbol{h}) \in X$, a solution $(\boldsymbol{u}, \boldsymbol{H}) \in X$ of the problem (2.5) exists and is unique; furthermore, with allowance for (2.1), the following estimate holds:

$$\|(\boldsymbol{u}, \boldsymbol{H})\|_1 \equiv (\|\boldsymbol{u}\|_1^2 + \|\boldsymbol{H}\|_1^2)^{1/2} \leqslant M/\lambda_* = (\|\boldsymbol{f}\| + \mu\|\boldsymbol{E}_0\|)/\lambda_*. \tag{2.7}$$

In the space X, we introduce a sphere $B_r = \{(\boldsymbol{v}, \boldsymbol{\Psi}) \in X\colon \|(\boldsymbol{v}, \boldsymbol{\Psi})\|_1 \leqslant r\}$, where $r = M/\lambda_*$. From the construction of the sphere B_r, it follows that the operator G maps B_r into itself. It is easy to show that G is compact and continuous. In this case, Schauder's theorem implies that the operator G has a fixed point $(\boldsymbol{u}, \boldsymbol{H}) = G(\boldsymbol{u}, \boldsymbol{H}) \in X$. The indicated point $(\boldsymbol{u}, \boldsymbol{H})$ is the desired solution of the problem (2.4). Let us formulate the result obtained.

Theorem 2.1. *If Conditions* 1–3 *are satisfied, at least one solution* $(\boldsymbol{u}, \boldsymbol{H}) \in X$ *of the problem* (2.4) *exists and the estimate* (2.7) *holds for this solution.*

Let us establish sufficient conditions of the uniqueness of the weak solution of the Problem 1.

Theorem 2.2. Let *Conditions* 1–3 *be satisfied. Then, there exists not more than one weak solution of the Problem* 1 *that satisfies the conditions*

$$\|\boldsymbol{u}\|_1 + \frac{\gamma_1}{\gamma}\frac{\sqrt{\mu}}{2}\|\boldsymbol{H}\|_1 < \frac{\alpha_0 \nu}{\gamma}, \qquad \|\boldsymbol{u}\|_1 + \frac{\sqrt{\mu}}{2}\|\boldsymbol{H}\|_1 < \frac{\alpha_1 \nu_m}{\gamma_1 \mu}. \tag{2.8}$$

Proof. We assume that the problem 1 has two solutions $(\boldsymbol{u}_1, \boldsymbol{H}_1)$ and $(\boldsymbol{u}_2, \boldsymbol{H}_2)$. Then, their difference ($\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2 \in \boldsymbol{V}$ and $\boldsymbol{H} = \boldsymbol{H}_1 - \boldsymbol{H}_2 \in \boldsymbol{V}_T$) satisfies the condition

$$\nu a_0(\boldsymbol{u}, \boldsymbol{u}) + \nu_1 a_1(\boldsymbol{H}, \boldsymbol{H}) + c(\boldsymbol{u}, \boldsymbol{u}_1, \boldsymbol{u}) + \mu c_1(\boldsymbol{H}, \boldsymbol{H}, \boldsymbol{u}_1) - \mu c_1(\boldsymbol{H}_1, \boldsymbol{H}, \boldsymbol{u}) = 0. \tag{2.9}$$

By virtue of (1.8) and (1.9) the following estimates are valid:

$$\begin{gathered} |c(\boldsymbol{u}, \boldsymbol{u}_1, \boldsymbol{u})| \leqslant \gamma\|\boldsymbol{u}_1\|_1\|\boldsymbol{u}\|_1^2, \qquad \mu|c_1(\boldsymbol{H}, \boldsymbol{H}, \boldsymbol{u}_1)| \leqslant \gamma_1\mu\|\boldsymbol{u}_1\|_1\|\boldsymbol{H}\|_1^2, \\ \mu|c_1(\boldsymbol{H}_1, \boldsymbol{H}, \boldsymbol{u})| \leqslant \gamma_1\mu\|\boldsymbol{H}_1\|_1\|\boldsymbol{H}\|_1\|\boldsymbol{u}\|_1 \leqslant \gamma_1\sqrt{\mu}\|\boldsymbol{H}_1\|_1(\mu\|\boldsymbol{H}\|_1^2 + \|\boldsymbol{u}\|_1^2)/2. \end{gathered} \tag{2.10}$$

Using (2.10) with allowance for (1.6) and (1.7), from (2.9) we obtain

$$(\alpha_0\nu - \gamma\|\boldsymbol{u}_1\|_1 - \gamma_1\sqrt{\mu}\|\boldsymbol{H}_1\|_1/2)\|\boldsymbol{u}\|_1^2 + (\alpha_1\nu_1 - \gamma_1\mu\|\boldsymbol{u}_1\|_1 - \gamma_1\mu\sqrt{\mu}\|\boldsymbol{H}_1\|_1/2)\|\boldsymbol{H}\|_1^2 \leqslant 0. \tag{2.11}$$

Assuming that the solution $(\boldsymbol{u}_1, \boldsymbol{H}_1)$ satisfies conditions (2.8), from (2.11) we obtain $\boldsymbol{u} = 0$ and $\boldsymbol{H} = 0$ and, hence, $\boldsymbol{u}_1 = \boldsymbol{u}_2$ and $\boldsymbol{H}_1 = \boldsymbol{H}_2$. We note that by virtue of Theorem 2.1, condition (2.8) is necessarily satisfied if the initial data $\boldsymbol{f}$ and $\boldsymbol{E}_0$ are "small" in the sense that $(1 + æ\sqrt{\mu})(\|\boldsymbol{f}\| + \mu\|\boldsymbol{E}_0\|) \leqslant \lambda_* \min(\alpha_0\nu/\gamma, \alpha_1\nu_m/\gamma_1)$, where $æ = (1/2)\max(1, \gamma_1/\gamma)$.

3. Formulation and Examination of the Control Problem. The current density $\boldsymbol{E}_0$ is further considered the required control and is designated by $\boldsymbol{g}$ ($\boldsymbol{g}$ varies in a certain set K). We assume that for K the following condition hold.

Condition 4. $K \subset \boldsymbol{L}^2(\Omega)$ is a nonempty closed convex set.

To formulate the control problem, we introduce a quality functional of the form $J(\boldsymbol{x},\boldsymbol{g}) = \tilde{J}(\boldsymbol{x}) + \alpha\ |\boldsymbol{g}\ |^2/2$ ($\alpha = \mathrm{const}$). Here $\tilde{J}\colon X \to \mathbb{R}$ is a functional that is weakly semicontinuous from below. In addition to Condition 4, we assume that the following condition holds:

Condition 5. $\alpha \geqslant 0$ and K is a limited set or $\alpha > 0$ and the functional $\tilde{J}$ is bounded from below.

Considering the functional J on weak solutions of the Problem 1, we write the main constraint (2.4) between the state $\boldsymbol{x} = (\boldsymbol{u},\boldsymbol{H})$ and the control $\boldsymbol{g}$:

$$F(\boldsymbol{x},\boldsymbol{g}) \equiv F(\boldsymbol{u},\boldsymbol{H},\boldsymbol{g}) = 0. \tag{3.1}$$

Here F is an operator that acts from $X \times K$ in X^* by the formula

$$\langle F(\boldsymbol{x},\boldsymbol{g}),(\boldsymbol{v},\boldsymbol{\Psi})\rangle = a((\boldsymbol{u},\boldsymbol{H}),(\boldsymbol{v},\boldsymbol{\Psi})) + c(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + \mu[c_1(\boldsymbol{\Psi},\boldsymbol{H},\boldsymbol{u}) - c_1(\boldsymbol{H},\boldsymbol{H},\boldsymbol{v})] - \langle \boldsymbol{F},(\boldsymbol{v},\boldsymbol{\Psi})\rangle.$$

We consider the problem

$$J(\boldsymbol{x},\boldsymbol{g}) \equiv J(\boldsymbol{u},\boldsymbol{H},\boldsymbol{g}) \to \inf, \qquad F(\boldsymbol{x},\boldsymbol{g}) = 0, \qquad (\boldsymbol{x},\boldsymbol{g}) \in X \times K. \tag{3.2}$$

As the possible quality functionals, we consider the following:

$$J_1(\boldsymbol{x}) = \frac{1}{2}\int\limits_\Omega |\operatorname{rot}\boldsymbol{u}|^2\,d\Omega, \quad J_2(\boldsymbol{x}) = \frac{1}{2}\int\limits_\Omega |\boldsymbol{u}-\boldsymbol{u}_d|^2\,d\Omega, \quad J_3(\boldsymbol{x}) = \frac{1}{2}\int\limits_\Omega |\boldsymbol{H}-\boldsymbol{H}_d|^2\,d\Omega. \tag{3.3}$$

Here $\boldsymbol{u}_d \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}_d \in \boldsymbol{L}^2(\Omega)$ are specified functions. (For the functionals J_k see [8, 11].) We set $Z_{ad} = \{(\boldsymbol{x},\boldsymbol{g}) \in X \times K\colon F(\boldsymbol{x},\boldsymbol{g}) = 0,\ J(\boldsymbol{x},\boldsymbol{g}) < \infty\}$.

Theorem 3.1. Let *Conditions* 1, 2, 4, and 5 *be satisfied,* $\tilde{J}\colon X \to \mathbb{R}$ *be a functional that is weakly semicontinuous from below, and the set* Z_{ad} *be not empty. Then there exists at least one solution of the problem* (3.2).

Proof. We use $(\boldsymbol{x}_m,\boldsymbol{g}_m) \equiv (\boldsymbol{u}_m,\boldsymbol{H}_m,\boldsymbol{g}_m) \in Z_{ad}$ to designate the minimizing sequence for which $\lim\limits_{m\to\infty} J(\boldsymbol{x}_m,\boldsymbol{g}_m) = \inf\limits_{(\boldsymbol{x}_m,\boldsymbol{g}_m)\in Z_{ad}} J(\boldsymbol{x}_m,\boldsymbol{g}_m) \equiv J^*$. By virtue of Condition 5 and Theorem 2.1, for the controls $\boldsymbol{g}_m$ and solutions $(\boldsymbol{u}_m,\boldsymbol{H}_m)$ of the Problem 1, the estimates $\|\boldsymbol{g}_m\| \leqslant c_1$, $\|\boldsymbol{u}_m\|_1 \leqslant c_2$, and $\|\boldsymbol{H}_m\|_1 \leqslant c_3$ hold; the constants c_1, c_2, and c_3 do not depend on m. From this it follows that there exist weak limits $\boldsymbol{g}^* \in K$, $\boldsymbol{u}^* \in \boldsymbol{V}$, and $\boldsymbol{H}^* \in \boldsymbol{V}_T$ of some subsequences of the sequences $\{\boldsymbol{g}_m\}$, $\{\boldsymbol{u}_m\}$, and $\{\boldsymbol{H}_m\}$. Reasoning as in [5], we obtain $F(\boldsymbol{u}^*,\boldsymbol{H}^*,\boldsymbol{g}^*) = 0$; in this case, because the functional J is weakly semicontinuous from bellow, it follows that $J(\boldsymbol{u}^*,\boldsymbol{H}^*,\boldsymbol{g}^*) = J^*$.

We note that each of the functionals J_1, J_2, and J_3 is nonnegative, is bounded from below, and is weakly semicontinuous from below. This fact and theorem 3.1 imply the following theorem:

Theorem 3.2. *If the conditions of Theorem* 3.1 *are satisfied, the extremum problem* (3.2) *has at least one solution for* $\tilde{J} = J_k$ ($k = 1,2,3$).

Let us prove the possibility of using the principle of undetermined Lagrangian multipliers for the extremum problem (3.2). As in [4–6], we employ the extremum principle in smoothly convex problems of conditional minimization [7]. We first calculate the partial Frechet derivative $F'_{\boldsymbol{x}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}})\colon X \to X^*$ of the operator F. From the definition of the derivative, it follows that at any point $(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}}) \equiv (\hat{\boldsymbol{u}},\hat{\boldsymbol{H}},\hat{\boldsymbol{g}}) \in X \times K$, the derivative $F'_{\boldsymbol{x}}(\hat{\boldsymbol{x}},\boldsymbol{g})$ is a linear continuous operator that sets a correspondence between each element $(\boldsymbol{w},\boldsymbol{h}) \in X$ and the functional $F'_{\boldsymbol{x}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}})(\boldsymbol{w},\boldsymbol{h}) \in X^*$ which acts by the formula

$$\langle F'_{\boldsymbol{x}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}})(\boldsymbol{w},\boldsymbol{h}),(\boldsymbol{v},\boldsymbol{\Psi})\rangle = \nu a_0(\boldsymbol{w},\boldsymbol{v}) + \nu_1 a_1(\boldsymbol{h},\boldsymbol{\Psi}) + [c(\hat{\boldsymbol{u}},\boldsymbol{w},\boldsymbol{v}) + c(\boldsymbol{w},\hat{\boldsymbol{u}},\boldsymbol{v})]$$

$$+ \mu[c_1(\boldsymbol{\Psi},\boldsymbol{h},\hat{\boldsymbol{u}}) + c_1(\boldsymbol{\Psi},\hat{\boldsymbol{H}},\boldsymbol{w})] - \mu[c_1(\hat{\boldsymbol{H}},\boldsymbol{h},\boldsymbol{v}) + c_1(\boldsymbol{h},\hat{\boldsymbol{H}},\boldsymbol{v})] \qquad \forall(\boldsymbol{v},\boldsymbol{\Psi}) \in X = \boldsymbol{V}\times\boldsymbol{V}_T. \tag{3.4}$$

A simple analysis shows that for all functionals in (3.3), the Frechet derivatives with respect to $\boldsymbol{u}$ and $\boldsymbol{H}$ at any point $\hat{\boldsymbol{x}} \in X$ exist and belong to the space X^*. In particular,

$$\langle (J_1)'_{\boldsymbol{u}}(\hat{\boldsymbol{x}}),\boldsymbol{w}\rangle = \int\limits_\Omega \operatorname{rot}\hat{\boldsymbol{u}}\cdot\operatorname{rot}\boldsymbol{w}\,d\Omega, \quad \langle (J_2)'_{\boldsymbol{u}}(\hat{\boldsymbol{x}}),\boldsymbol{w}\rangle = \int\limits_\Omega (\hat{\boldsymbol{u}}-\boldsymbol{u}_d)\cdot\boldsymbol{w}\,d\Omega \quad \forall \boldsymbol{w}\in\boldsymbol{V}. \tag{3.5}$$

Let us introduce a set $F(\hat{\boldsymbol{x}}, K) = F(\hat{\boldsymbol{u}}, \hat{\boldsymbol{H}}, K)$ which is a convex subset of the space X^*. We set $\boldsymbol{z} = (\boldsymbol{\xi}, \boldsymbol{\eta}) \in X$ and introduce a Lagrangian $\mathcal{L}$ for the functional J using the formula $\mathcal{L}(\boldsymbol{x}, \boldsymbol{g}, \lambda_0, \boldsymbol{z}) = \lambda_0 J(\boldsymbol{x}, \boldsymbol{g}) + \langle F(\boldsymbol{x}, \boldsymbol{g}), (\boldsymbol{\xi}, \boldsymbol{\eta})\rangle_{X^* \times X}$.

Theorem 3.3. *If Conditions* 1, 2, and 4 *are satisfied, let* $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{g}}) \equiv (\hat{\boldsymbol{u}}, \hat{\boldsymbol{H}}, \hat{\boldsymbol{g}}) \in X \times K$ *be a local minimum point in the problem* (3.2) *and let the functional* $J(\boldsymbol{x}, \boldsymbol{g})$ *be continuously differentiable with respect to* $\boldsymbol{x}$ *at the point* $\hat{\boldsymbol{x}}$ *for any element* $\boldsymbol{g} \in K$ *and be convex over* $\boldsymbol{g}$ *for each point* $\boldsymbol{x} \in X$. *Then there exists a nonzero Lagrangian multiplier* $(\lambda_0, \boldsymbol{z}) \equiv (\lambda_0, \boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^+ \times X$ *such that the Euler–Lagrange equation*

$$\lambda_0 \langle J'_{\boldsymbol{x}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{g}}), (\boldsymbol{w}, \boldsymbol{h})\rangle_{X^* \times X} + \langle F'_{\boldsymbol{x}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{g}})(\boldsymbol{w}, \boldsymbol{h}), (\boldsymbol{\xi}, \boldsymbol{\eta})\rangle_{X^* \times X} = 0 \qquad \forall (\boldsymbol{w}, \boldsymbol{h}) \in X \tag{3.6}$$

is valid and the minimum principle $\mathcal{L}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{g}}, \lambda_0, \boldsymbol{\xi}, \boldsymbol{\eta}) \leqslant \mathcal{L}(\hat{\boldsymbol{x}}, \boldsymbol{g}, \lambda_0, \boldsymbol{\xi}, \boldsymbol{\eta})$ holds for $\boldsymbol{g} \in K$ or

$$(\boldsymbol{g} - \hat{\boldsymbol{g}}, \boldsymbol{\eta}) \leqslant \lambda_0 [J(\hat{\boldsymbol{x}}, \boldsymbol{g}) - J(\hat{\boldsymbol{x}}, \hat{\boldsymbol{g}})] \qquad \forall \boldsymbol{g} \in K. \tag{3.7}$$

Proof. By virtue of theorem 3 in [7, chapter 1], to prove the existence of the Lagrangian multiplier $(\lambda_0, \boldsymbol{z})$ taking into account the differentiability of F on X for each $\boldsymbol{g} \in K$ and the convexity of the sets K and $F(\hat{\boldsymbol{x}}, K)$, one needs to show the operator $F'_{\boldsymbol{x}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{g}})$: $X \to X^*$ is a Fredholm one. By virtue of (3.4), we have $F'_{\boldsymbol{x}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{g}}) = \hat{F}_1 + \hat{F}_2$, where $\langle \hat{F}_1(\boldsymbol{w}, \boldsymbol{h}), (\boldsymbol{v}, \boldsymbol{\Psi})\rangle = a((\boldsymbol{w}, \boldsymbol{h}), (\boldsymbol{v}, \boldsymbol{\Psi})) + c(\hat{\boldsymbol{u}}, \boldsymbol{w}, \boldsymbol{v}) + \mu[c_1(\boldsymbol{\Psi}, \hat{\boldsymbol{H}}, \boldsymbol{w}) - c_1(\boldsymbol{h}, \hat{\boldsymbol{H}}, \boldsymbol{v})]$ and $\langle \hat{F}_2(\boldsymbol{w}, \boldsymbol{h}), (\boldsymbol{v}, \boldsymbol{\Psi})\rangle = c(\boldsymbol{w}, \hat{\boldsymbol{u}}, \boldsymbol{v}) + \mu[c_1(\boldsymbol{\Psi}, \boldsymbol{h}, \hat{\boldsymbol{u}}) - c_1(\hat{\boldsymbol{H}}, \boldsymbol{h}, \boldsymbol{v})]$. Obviously, the operator $\hat{F}_1$: $X \to X^*$ is linear, continuous and, moreover, coercive because, by virtue of (1.10) and (1.11), $\langle \hat{F}_1(\boldsymbol{v}, \boldsymbol{\Psi}), (\boldsymbol{v}, \boldsymbol{\Psi})\rangle = a((\boldsymbol{v}, \boldsymbol{\Psi})$ and $(\boldsymbol{v}, \boldsymbol{\Psi})) \geqslant \lambda_* \|(\boldsymbol{v}, \boldsymbol{\Psi})\|_1^2$. In addition, Eqs. (1.8) and (1.9) and the compactness of the embedding $\boldsymbol{H}^1(\Omega) \subset \boldsymbol{L}^4(\Omega)$ imply the continuity and compactness of the operator $\hat{F}_2$. From this it follows that the operator $F'_{\boldsymbol{x}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{g}}) = \hat{F}_1 + \hat{F}_2$ is a Fredholm one because it is the sum of an isomorphism and a continuous compact operator.

It is easy to see that each of the functionals J_1, J_2, and J_3 introduced above satisfies all conditions of Theorem 3.3. Moreover, the Euler–Lagrange equation (3.6) provides additional information on their properties. For this, taking into account Eq. (3.4) we write Eq. (3.6) in the form

$$\lambda_0 \langle J'_{\boldsymbol{x}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{g}}), (\boldsymbol{w}, \boldsymbol{h})\rangle_{X^* \times X} + \nu a_0(\boldsymbol{w}, \boldsymbol{\xi}) + \nu_1 a_1(\boldsymbol{h}, \boldsymbol{\eta}) + [c(\hat{\boldsymbol{u}}, \boldsymbol{w}, \boldsymbol{\xi}) + c(\boldsymbol{w}, \hat{\boldsymbol{u}}, \boldsymbol{\xi})]$$

$$+ \mu[c_1(\boldsymbol{\eta}, \boldsymbol{h}, \hat{\boldsymbol{u}}) + c_1(\boldsymbol{\eta}, \hat{\boldsymbol{H}}, \boldsymbol{w})] - \mu[c_1(\hat{\boldsymbol{H}}, \boldsymbol{h}, \boldsymbol{\xi}) + c_1(\boldsymbol{h}, \hat{\boldsymbol{H}}, \boldsymbol{\xi})] = 0 \qquad \forall (\boldsymbol{w}, \boldsymbol{h}) \in X. \tag{3.8}$$

First setting $\boldsymbol{h} = 0$ and then $\boldsymbol{w} = 0$ in (3.8), we arrive at the following identities:

$$\nu a_0(\boldsymbol{w}, \boldsymbol{\xi}) + c(\hat{\boldsymbol{u}}, \boldsymbol{w}, \boldsymbol{\xi}) + c(\boldsymbol{w}, \hat{\boldsymbol{u}}, \boldsymbol{\xi}) + \mu c_1(\boldsymbol{\eta}, \hat{\boldsymbol{H}}, \boldsymbol{w}) + \lambda_0 \langle J'_{\boldsymbol{u}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{g}}), \boldsymbol{w}\rangle = 0 \qquad \forall \boldsymbol{w} \in \boldsymbol{V}; \tag{3.9}$$

$$\nu_1 a_1(\boldsymbol{h}, \boldsymbol{\eta}) + \mu c_1(\boldsymbol{\eta}, \boldsymbol{h}, \hat{\boldsymbol{u}}) - \mu[c_1(\hat{\boldsymbol{H}}, \boldsymbol{h}, \boldsymbol{\xi}) + c_1(\boldsymbol{h}, \hat{\boldsymbol{H}}, \boldsymbol{\xi})] + \lambda_0 \langle J'_{\boldsymbol{H}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{g}}), \boldsymbol{h}\rangle = 0 \qquad \forall \boldsymbol{h} \in \boldsymbol{V}_T. \tag{3.10}$$

As a result, we obtain the optimality system for determining the required control $\hat{\boldsymbol{g}}$ and its corresponding "optimal" distributions of the velocity and magnetic field in the domain Ω. This system consists of three parts. The first part has the form of the weak formulation (2.4) of Problem 1, equivalent to (3.1), the second part includes the identities (3.9) and (3.10) for $(\boldsymbol{\xi}, \boldsymbol{\eta})$, and the last part is inequality (3.7).

Remark 3.1. We note that the first two parts of the optimality system are formally obtained by equating to zero the first derivatives (first variations) of the Lagrangian $\mathcal{L}$ with respect to the corresponding variables. In particular, (2.4) is obtained by equating to zero the first variation of the Lagrangian $\mathcal{L}$ over $(\boldsymbol{\xi}, \boldsymbol{\eta})$, whereas (3.9) and (3.10) are obtained by equating to zero the first variations over $\boldsymbol{u}$ and $\boldsymbol{H}$.

Let us show that system (3.9), (3.10) can be treated as a weak formulation of a certain boundary-value problem for Lagrange multipliers $(\boldsymbol{\xi}, \boldsymbol{\eta})$. Assuming for simplicity that Ω is a simply connected domain, so that $\mathcal{H}(m) = \{0\}$, and the embedding $\boldsymbol{V} \subset \boldsymbol{V}_T$ is valid, we introduce a linear operator $S_{\boldsymbol{V}}$: $\boldsymbol{V}_T^* \to \boldsymbol{V}^*$ which acts by the formula

$$\langle S_{\boldsymbol{V}} \boldsymbol{l}, \boldsymbol{h}\rangle_{\boldsymbol{V}^* \times \boldsymbol{V}} = \langle \boldsymbol{l}, \boldsymbol{h}\rangle_{\boldsymbol{V}_T^* \times \boldsymbol{V}_T} \qquad \forall \boldsymbol{l} \in \boldsymbol{V}_T^*, \quad \boldsymbol{h} \in \boldsymbol{V} \subset \boldsymbol{V}_T. \tag{3.11}$$

Equations (1.12)–(1.14), (1.10), and (1.5) lead to the relations

$$a_0(\boldsymbol{w}, \boldsymbol{\xi}) = -\langle \Delta \boldsymbol{\xi}, \boldsymbol{w}\rangle, \qquad b(\boldsymbol{w}, \sigma) = \langle \nabla \sigma, \boldsymbol{w}\rangle, \qquad a_1(\boldsymbol{h}, \boldsymbol{\eta}) = \langle \operatorname{rot} \operatorname{rot} \boldsymbol{\eta}, \boldsymbol{w}\rangle,$$

$$c(\hat{\boldsymbol{u}}, \boldsymbol{w}, \boldsymbol{\xi}) = -c(\hat{\boldsymbol{u}}, \boldsymbol{\xi}, \boldsymbol{w}) \equiv -((\hat{\boldsymbol{u}} \cdot \nabla)\boldsymbol{\xi}, \boldsymbol{w}), \qquad c(\boldsymbol{w}, \hat{\boldsymbol{u}}, \boldsymbol{\xi}) \equiv \int_\Omega (\boldsymbol{w} \cdot \nabla)\hat{\boldsymbol{u}} \cdot \boldsymbol{\xi}\, d\Omega = (\nabla \hat{\boldsymbol{u}}^{\mathrm{t}} \cdot \boldsymbol{\xi}, \boldsymbol{w}),$$

$$c_1(\boldsymbol{\eta}, \hat{\boldsymbol{H}}, \boldsymbol{w}) = (\operatorname{rot} \boldsymbol{\eta} \times \hat{\boldsymbol{H}}, \boldsymbol{w}), \quad c_1(\boldsymbol{\eta}, \boldsymbol{h}, \hat{\boldsymbol{u}}) = -(\operatorname{rot} \boldsymbol{\eta} \times \hat{\boldsymbol{u}}, \boldsymbol{h}), \tag{3.12}$$

$$c_1(\hat{\boldsymbol{H}},\boldsymbol{h},\boldsymbol{\xi}) = -(\mathrm{rot}\,\hat{\boldsymbol{H}}\times\boldsymbol{\xi},\boldsymbol{h}), \qquad c_1(\boldsymbol{h},\hat{\boldsymbol{H}},\boldsymbol{\xi}) \equiv (\mathrm{rot}\,\boldsymbol{h},\hat{\boldsymbol{H}}\times\boldsymbol{\xi}) = (\mathrm{rot}(\hat{\boldsymbol{H}}\times\boldsymbol{\xi}),\boldsymbol{h}) \quad \forall(\boldsymbol{w},\boldsymbol{h})\in\boldsymbol{H}_0^1(\Omega)^2,$$

where $\nabla\hat{\boldsymbol{u}}^{\mathrm{t}}$ is a tensor conjugate to $\nabla\hat{\boldsymbol{u}}$. Using these relations, we obtain

$$\nu a_0(\boldsymbol{w},\boldsymbol{\xi}) + c(\hat{\boldsymbol{u}},\boldsymbol{w},\boldsymbol{\xi}) + c(\boldsymbol{w},\hat{\boldsymbol{u}},\boldsymbol{\xi}) + \mu c_1(\boldsymbol{\eta},\hat{\boldsymbol{H}},\boldsymbol{w}) = -\nu\langle\Delta\boldsymbol{\xi},\boldsymbol{w}\rangle$$

$$-((\hat{\boldsymbol{u}}\cdot\nabla)\boldsymbol{\xi},\boldsymbol{w}) + (\nabla\hat{\boldsymbol{u}}^{\mathrm{t}}\cdot\boldsymbol{\xi},\boldsymbol{w}) + \mu(\mathrm{rot}\,\boldsymbol{\eta}\times\hat{\boldsymbol{H}},\boldsymbol{w}) \qquad \forall\boldsymbol{w}\in\boldsymbol{H}_0^1(\Omega); \tag{3.13}$$

$$\nu_1 a_1(\boldsymbol{h},\boldsymbol{\eta}) + \mu c_1(\boldsymbol{\eta},\boldsymbol{h},\hat{\boldsymbol{u}}) - \mu[c_1(\hat{\boldsymbol{H}},\boldsymbol{h},\boldsymbol{\xi}) + c_1(\boldsymbol{h},\hat{\boldsymbol{H}},\boldsymbol{\xi})] = \nu_m\langle\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\eta},\boldsymbol{h}\rangle$$

$$-\mu(\mathrm{rot}\,\boldsymbol{\eta}\times\hat{\boldsymbol{u}},\boldsymbol{h}) + \mu(\mathrm{rot}\,\hat{\boldsymbol{H}}\times\boldsymbol{\xi},\boldsymbol{h}) - \mu(\mathrm{rot}(\hat{\boldsymbol{H}}\times\boldsymbol{\xi}),\boldsymbol{h}) \qquad \forall\boldsymbol{h}\in\boldsymbol{H}_0^1(\Omega). \tag{3.14}$$

Assuming that the Frechet derivatives of the functional J satisfy the conditions

$$J'_{\boldsymbol{u}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}})\in\boldsymbol{H}^{-1}(\Omega), \qquad S_{\boldsymbol{V}}J'_{\boldsymbol{H}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}})\in\boldsymbol{H}^{-1}(\Omega), \tag{3.15}$$

we introduce the following functionals in $\boldsymbol{H}_0^1(\Omega)$: $\boldsymbol{L}_1 = -\nu\Delta\boldsymbol{\xi} - (\hat{\boldsymbol{u}}\cdot\nabla)\boldsymbol{\xi} + \nabla\hat{\boldsymbol{u}}^{\mathrm{t}}\cdot\boldsymbol{\xi} + \mu\,\mathrm{rot}\,\boldsymbol{\eta}\times\hat{\boldsymbol{H}} + \lambda_0 J'_{\boldsymbol{u}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}})$ and $\boldsymbol{L}_2 = \nu_1\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\eta} - \mu\,\mathrm{rot}\,\boldsymbol{\eta}\times\hat{\boldsymbol{u}} + \mu\,\mathrm{rot}\,\hat{\boldsymbol{H}}\times\boldsymbol{\xi} - \mu\,\mathrm{rot}\,(\hat{\boldsymbol{H}}\times\boldsymbol{\xi}) + \lambda_0 S_{\boldsymbol{V}}J'_{\boldsymbol{H}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}})$. From (3.13) and (3.14) it follows that the identity (3.9) and the restriction of the identity (3.10) on $\boldsymbol{V}$ can be written as $\langle\boldsymbol{L}_1,\boldsymbol{w}\rangle = 0\ \forall\boldsymbol{w}\in\boldsymbol{V}$ and $\langle\boldsymbol{L}_2,\boldsymbol{h}\rangle = 0\ \forall\boldsymbol{h}\in\boldsymbol{V}$. Since, by virtue of (3.15), $\boldsymbol{L}_i\in\boldsymbol{H}^{-1}(\Omega)$, from [10, p. 22] it follows that there exist functions $\sigma\in L_0^2(\Omega)$ and $\psi\in L_0^2(\Omega)$ such that

$$-\nu\Delta\boldsymbol{\xi} - (\hat{\boldsymbol{u}}\cdot\nabla)\boldsymbol{\xi} + \nabla\hat{\boldsymbol{u}}^{\mathrm{t}}\cdot\boldsymbol{\xi} + \mu\,\mathrm{rot}\,\boldsymbol{\eta}\times\hat{\boldsymbol{H}} + \nabla\sigma = -\lambda_0 J'_{\boldsymbol{u}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}}) \quad \text{in} \quad \boldsymbol{H}^{-1}(\Omega); \tag{3.16}$$

$$\nu_1\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\eta} - \mu\,\mathrm{rot}\,\boldsymbol{\eta}\times\hat{\boldsymbol{u}} + \mu\,\mathrm{rot}\,\hat{\boldsymbol{H}}\times\boldsymbol{\xi} - \mu\,\mathrm{rot}\,(\hat{\boldsymbol{H}}\times\boldsymbol{\xi}) + \nabla\psi = -\lambda_0 S_{\boldsymbol{V}}J'_{\boldsymbol{H}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}}) \quad \text{in} \quad \boldsymbol{H}^{-1}(\Omega). \tag{3.17}$$

Let us formulate the result obtained.

Theorem 3.4. *Let Ω be a simply connected domain and the conditions of Theorem* 3.3 *and* (3.15) *be satisfied. Then, there exist functions $(\boldsymbol{\xi},\boldsymbol{\eta})\in\boldsymbol{V}\times\boldsymbol{V}_T$, $\sigma\in L_0^2(\Omega)$ and $\psi\in L_0^2(\Omega)$ and a constant $\lambda_0\geqslant 0$ such that they, together with the element $(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}}) = (\hat{\boldsymbol{u}},\hat{\boldsymbol{H}},\hat{\boldsymbol{g}})$, fit Eqs.* (3.16) and (3.17), *the identities* (3.9) and (3.10), *and the minimum principle* (3.7).

Remark 3.2. Although the functions σ and ψ formally do not enter into the expression for $\mathcal{L}$, they can be considered Lagrangian multipliers that are conjugate to the pressure p and the electric field potential $\boldsymbol{E}$ included in the model (1.1), (1.2).

We assume that the Lagrange multipliers $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, σ, and ψ, as well as the functions $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{H}}$, possess additional smoothness; namely, $(\boldsymbol{\xi},\boldsymbol{\eta},\hat{\boldsymbol{u}},\hat{\boldsymbol{H}})\in\boldsymbol{H}^1(\Delta;\Omega)$, $(\sigma,\psi)\in H^1(\Omega)$. Then, from (3.9), (3.10) and (3.16), (3.17), one can derive "pointwise" differential equations and boundary relations for $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, σ, and ψ. Indeed, if these conditions are satisfied, the left side in (3.17), as well as the right side, is a function that belongs to the space $\boldsymbol{L}^{3/2}(\Omega)$, and $\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\eta}\equiv\Delta\boldsymbol{\eta}\in\boldsymbol{L}^2(\Omega)$. In view of this, it is possible to multiply (3.17) by the function $\boldsymbol{h}\in\boldsymbol{V}_T\subset\boldsymbol{L}^6(\Omega)$ and integrate over the domain Ω. Applying the Green formula (1.13) for $\boldsymbol{q} = \mathrm{rot}\,\boldsymbol{\eta}\in\boldsymbol{H}(\mathrm{rot};\Omega)$ and taking into account (3.12), we arrive at the identity

$$\nu_1 a_1(\boldsymbol{\eta},\boldsymbol{h}) - \nu_1\langle\mathrm{rot}\,\boldsymbol{\eta}\times\boldsymbol{n},\boldsymbol{h}\rangle_\Gamma + \mu c_1(\boldsymbol{\eta},\boldsymbol{h},\hat{\boldsymbol{u}}) - \mu c_1(\hat{\boldsymbol{H}},\boldsymbol{h},\boldsymbol{\xi}) - \mu c_1(\boldsymbol{h},\hat{\boldsymbol{H}},\boldsymbol{\xi})$$

$$= -\lambda_0\int_\Omega S_{\boldsymbol{V}}J'_{\boldsymbol{H}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}})\cdot\boldsymbol{h}\,d\Omega \qquad \forall\boldsymbol{h}\in\boldsymbol{V}_T. \tag{3.18}$$

Subtracting the identity (3.18) from the identity (3.10), we obtain

$$\nu_1\langle\mathrm{rot}\,\boldsymbol{\eta}\times\boldsymbol{n},\boldsymbol{h}\rangle_\Gamma = \lambda_0\int_\Omega S_{\boldsymbol{V}}J'_{\boldsymbol{H}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}})\cdot\boldsymbol{h}\,d\Omega - \lambda_0\langle J'_{\boldsymbol{H}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}}),\boldsymbol{h}\rangle \qquad \forall\boldsymbol{h}\in\boldsymbol{V}_T. \tag{3.19}$$

Let us consider the case $\tilde{J} = J_1$ [J_1 is defined by the first formula in (3.3)]. A simple analysis taking into account (3.5) and (1.13) shows that

$$\langle(J_1)'_{\boldsymbol{u}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}}),\boldsymbol{w}\rangle = \int_\Omega \mathrm{rot}\,\hat{\boldsymbol{u}}\cdot\mathrm{rot}\,\boldsymbol{w}\,d\Omega = \int_\Omega \mathrm{rot}\,\mathrm{rot}\,\hat{\boldsymbol{u}}\cdot\boldsymbol{w}\,d\Omega \quad \forall\boldsymbol{w}\in\boldsymbol{V},$$

$$(J_1)'_{\boldsymbol{H}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}}) = 0. \tag{3.20}$$

Using (3.20), we write (3.16), (3.17), and (3.19) as

$$-\nu\Delta\boldsymbol{\xi} - (\hat{\boldsymbol{u}}\cdot\nabla)\boldsymbol{\xi} + \nabla\hat{\boldsymbol{u}}^{\mathrm{t}}\cdot\boldsymbol{\xi} + \mu\,\mathrm{rot}\,\boldsymbol{\eta}\times\hat{\boldsymbol{H}} + \nabla\sigma = -\lambda_0\,\mathrm{rot}\,\mathrm{rot}\,\hat{\boldsymbol{u}} \quad \text{in} \quad \Omega; \tag{3.21}$$

$$\nu_1\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\eta} - \mu\,\mathrm{rot}\,\boldsymbol{\eta}\times\hat{\boldsymbol{u}} + \mu\,\mathrm{rot}\,\hat{\boldsymbol{H}}\times\boldsymbol{\xi} - \mu\,\mathrm{rot}\,(\hat{\boldsymbol{H}}\times\boldsymbol{\xi}) + \nabla\psi = 0 \quad \text{in} \quad \Omega; \tag{3.22}$$

$$\nu_1\langle\mathrm{rot}\,\boldsymbol{\eta}\times\boldsymbol{n},\boldsymbol{h}\rangle_\Gamma = 0 \qquad \forall\boldsymbol{h}\in\boldsymbol{V}_T. \tag{3.23}$$

Equations (3.21) and (3.22), together with the relations

$$\mathrm{div}\,\boldsymbol{\xi} = 0,\ \mathrm{div}\,\boldsymbol{\eta} = 0 \text{ in } \Omega, \quad \boldsymbol{\xi} = 0,\ \boldsymbol{\eta}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma, \quad \mathrm{rot}\,\boldsymbol{\eta}\times\boldsymbol{n} = 0 \text{ on } \Gamma \tag{3.24}$$

resulting from the conditions $\boldsymbol{\xi}\in\boldsymbol{V}$ and $\boldsymbol{\eta}\in\boldsymbol{V}_T$ and (3.23), characterize the second part of the optimality system for the problem (3.2) for $\tilde{J} = J_1$ if condition (3.15) is satisfied.

If the form of the functional $\tilde{J}$ changes, some of the relations given above also change. In particular, for the functional J_2, which, like J_1, depends only on $\boldsymbol{u}$, Eq. (3.21) for the multiplier $\boldsymbol{\xi}$ changes and, with allowance for (3.5), it becomes

$$-\nu\Delta\boldsymbol{\xi} - (\hat{\boldsymbol{u}}\cdot\nabla)\boldsymbol{\xi} + \nabla\hat{\boldsymbol{u}}^{\mathrm{t}}\cdot\boldsymbol{\xi} + \mu\,\mathrm{rot}\,\boldsymbol{\eta}\times\hat{\boldsymbol{H}} + \nabla\sigma = -\lambda_0(\hat{\boldsymbol{u}} - \boldsymbol{u}_d) \qquad \text{in} \quad \Omega, \tag{3.25}$$

whereas relations (3.22) and (3.24) remain unchanged. Similarly, for the functional J_3, which depends only on $\boldsymbol{H}$, we have $(J_3)'_{\boldsymbol{u}} = 0$,

$$\langle (J_3)'_{\boldsymbol{H}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}}),\boldsymbol{h}\rangle = \int\limits_\Omega (\hat{\boldsymbol{H}} - \boldsymbol{H}_d)\cdot\boldsymbol{h}\,d\Omega \quad \forall\boldsymbol{h}\in\boldsymbol{V}_T,$$

$$S_{\boldsymbol{V}}[(J_3)'_{\boldsymbol{H}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{g}})] = \hat{\boldsymbol{H}} - \boldsymbol{H}_d \in \boldsymbol{L}^2(\Omega). \tag{3.26}$$

Taking (3.26) into account, from (3.16), (3.17), and (3.19), we obtain relations (3.24) and the equation

$$-\nu\Delta\boldsymbol{\xi} - (\hat{\boldsymbol{u}}\cdot\nabla)\boldsymbol{\xi} + \nabla\hat{\boldsymbol{u}}^{\mathrm{t}}\cdot\boldsymbol{\xi} + \mu\,\mathrm{rot}\,\boldsymbol{\eta}\times\hat{\boldsymbol{H}} + \nabla\sigma = 0 \qquad \text{in} \quad \Omega; \tag{3.27}$$

$$\nu_1\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\eta} - \mu\,\mathrm{rot}\,\boldsymbol{\eta}\times\hat{\boldsymbol{u}} + \mu\,\mathrm{rot}\,\hat{\boldsymbol{H}}\times\boldsymbol{\xi} - \mu\,\mathrm{rot}\,(\hat{\boldsymbol{H}}\times\boldsymbol{\xi}) + \nabla\psi = -\lambda_0(\hat{\boldsymbol{H}} - \boldsymbol{H}_d) \quad \text{in } \Omega. \tag{3.28}$$

If the functions $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{H}}$ are known, relations (3.21), (3.22), and (3.24), or (3.25), (3.22), and (3.24), or (3.27), (3.28), and (3.24) form a closed system of linear equations for the Lagrangian multipliers $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, σ, and ψ that is equivalent, by virtue of theorem 3.3, to the linear Fredholm problem. In the general case where $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{H}}$ are unknown, the indicated relations represent the second part of the optimality system that corresponds to the control problem (3.2) for the functionals J_1, J_2, and J_3. These relations should be considered together with the identity (2.4) or (3.1), which form the first part of the optimality system, and inequality (3.7). Obviously, solving the optimality system obtained is a rather complicated problem which requires developing effective numerical algorithms.

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